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On Birkhoff's theorem for electromagnetic fields in a scalar–tensor theory of gravitation

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Abstract. It is shown that an analogue of Birkhoff's theorem of general relativity exists for electromagnetic fields in a scalar–tensor theory of gravitation proposed by Sen and Dunn when the scalar field introduced in the theory is independent of time.

1. Introduction

Recently Sen and Dunn (1971) have proposed a new scalar–tensor theory of gravitation in which both the scalar and tensor fields have intrinsic geometrical significance. The scalar field in this theory is characterized by the function $\phi = \phi(x^i)$ where x^i are coordinates in the four-dimensional Lyra manifold and the tensor field is identified with the metric tensor g_{ij} of the manifold. The field equations given by Sen and Dunn for the combined scalar–tensor fields are

$$R_{ij} - \tfrac{1}{2}g_{ij}R = \omega\phi^{-2}(\phi_{,i}\phi_{,j} - \tfrac{1}{2}g_{ij}\phi_{,k}\phi^{,k}) - \phi^{-2}T_{ij} \tag{1}$$

where $\omega = \frac{3}{2}$, T_{ij} is the energy–momentum tensor of the field and R_{ij} and R are, respectively, the usual Ricci tensors and Riemann curvature scalar (in our units $c = 8\pi G = 1$). Reddy (1973) has shown that unlike in the Brans–Dicke scalar–tensor theory (Brans and Dicke 1961), Birkhoff's theorem of general relativity is valid in the present theory irrespective of the nature of the scalar field introduced in the theory.

In this paper we have shown, following Das (1960), that Birkhoff's theorem of general relativity for electromagnetic fields exists in the scalar–tensor theory proposed by Sen and Dunn when the scalar field introduced in the theory is independent of time.

2. Birkhoff's theorem in Sen–Dunn theory

It was shown by Birkhoff (1927) that every spherically symmetric solution of the Einstein vacuum field equations is static. This fact is known as Birkhoff's theorem. Shücking (1957) has shown that this theorem is valid in Jordan's (1952) extended theory of gravitation when the gravitational invariant of the theory is independent of time. Reddy (1973) has shown that Birkhoff's theorem holds in the Brans–Dicke theory of gravitation when the scalar field introduced in the theory is independent of time. It is also shown, therein, that Birkhoff's theorem is valid in the present theory irrespective of the nature of the scalar field introduced. Das (1960) has extended

Birkhoff's theorem in general relativity from the purely gravitational case to the combined electromagnetic and gravitational case. On similar lines we show, here, that Birkhoff's theorem exists in the Sen–Dunn theory of gravitation when the scalar field introduced in the theory is independent of time.

In the notation of Das (1960), the combined Sen–Dunn–Maxwell field equations in the absence of matter are

$$\begin{aligned} D^j &\equiv F^{ij}_{;j} = 0 \\ E^i &\equiv \tfrac{1}{2}\epsilon^{jkli} F_{kl,j} = F_{[kl,j]} = 0 \\ Q^i_j &\equiv R^i_j - \tfrac{1}{2}\delta^i_j R - (F^{ik}F_{jk} - \tfrac{1}{4}\delta^i_j F^{kl}F_{kj})\phi^{-2} - \omega(\phi)^{-2}(g^{ij}\phi_{,i}\phi_{,j} - \tfrac{1}{2}\delta^i_j g^{kl}\phi_{,k}\phi_{,l}) = 0 \end{aligned} \tag{2}$$

where commas and semicolons denote partial and covariant derivatives respectively.

We consider the spherically symmetric metric in the form

$$ds^2 = e^\beta\, dt^2 - e^\alpha\, dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\Phi^2) \tag{3}$$

where $\alpha = \alpha(r, t)$, $\beta = \beta(r, t)$, with the scalar field $\phi = \phi(r, t)$. From considerations of spherical symmetry we retain only the radial components of the electromagnetic field and (Das 1960)

$$F_{12} = F_{13} = F_{24} = F_{34} = 0. \tag{4}$$

Without this choice, the transverse components would define a physically distinguishable direction on the surface of a sphere which would destroy the spherical symmetry of the field. Demanding no other symmetry conditions on the surviving components F_{14} and F_{23} and in view of the choice (4) the electromagnetic field equations

$$D^1 = D^4 = E^2 = E^3 = 0 \qquad \text{and} \qquad D^2 = D^3 = E^1 = E^4 = 0 \tag{5}$$

yield

$$F_{14} = (\epsilon/r^2)\, e^{(\alpha+\beta)/2} \qquad F_{23} = \mu \sin\theta \tag{6}$$

where ϵ and μ are constants of integration which can be interpreted as the electric charge and the magnetic pole strength, respectively, of the point source. It was possible to include the magnetic contribution because we worked directly with the field strengths F_{14} and F_{23} instead of potentials and did not assume that they have any spatial symmetry.

Now using (4), (5) and (6) we can write down the Sen–Dunn–Maxwell field equations (2) for the metric (3) as

$$\begin{aligned} Q^1_1 + Q^4_4 &= \{1 + \tfrac{1}{2}r(\beta' - \alpha') - e^\alpha[1 - (\epsilon^2 + \mu^2)/r^2\phi^2]\} = 0 \\ Q^2_2 = Q^3_3 &\equiv e^{-\alpha}\left(\frac{\beta''}{2} + \frac{\beta'^2}{4} - \frac{\alpha'\beta'}{4} + \frac{\beta' - \alpha'}{2r}\right) - e^{-\beta}\left(\frac{\ddot{\alpha}}{2} + \frac{\dot{\alpha}^2}{4} - \frac{\dot{\alpha}\dot{\beta}}{4}\right) \\ &\quad - \tfrac{1}{2}\omega[e^{-\alpha}(\phi'/\phi)^2 - e^{-\beta}(\dot{\phi}/\phi)^2] - (\epsilon^2 + \mu^2)/\phi^2 r^4 = 0 \\ Q^1_4 &\equiv e^{-\alpha}\, \dot{\alpha}/r = 0 \end{aligned}$$

where primes denote partial differentiation with respect to r and supercript dots denote partial differentiation with respect to time t.

It can be seen that when ϕ is a constant the above system reduces to the Einstein–Maxwell field equations in the spherically symmetric case and hence Birkhoff's theorem follows as shown by Das (1960).

When the scalar field ϕ is a function of r alone, that is

$$\dot{\phi}=0 \tag{7}$$

we have from $Q_4^1=0$

$$\dot{\alpha}=0 \tag{8}$$

that is, α is independent of time.

From $Q_1^1+Q_4^4=0$ we have

$$1+\tfrac{1}{2}r(\beta'-\alpha')=\mathrm{e}^{\alpha}[1-(\epsilon^2+\mu^2)/r^2\phi^2]. \tag{9}$$

Differentiating (9) partially with respect to t and using (7) and (8) we get

$$\dot{\beta}'=0. \tag{10}$$

That is, β is linearly separable in t and r so that

$$\beta=f(r)+g(t)$$

where f and g are arbitrary functions of r and t respectively. Introducing a time transformation (Das 1960)

$$\mathrm{d}t'=\mathrm{e}^{g(t)/2}\,\mathrm{d}t$$

and dropping primes afterwards, it follows, in view of (8), that the metric (3) is static. Hence Birkhoff's theorem is valid in this theory when the scalar field introduced is independent of time.

When ϕ is a function of both r and t Birkhoff's theorem is not valid in this case since it can be seen from equation (9) that there is explicit scalar field interaction.

By a straightforward calculation it should not be difficult to show on similar lines, that Birkhoff's theorem for electromagnetic fields is also valid in the Brans–Dicke scalar–tensor theory of gravitation only when the scalar field introduced in the theory is independent of time.

3. Conclusions

We have shown that Birkhoff's theorem of general relativity is true for electromagnetic fields in a scalar–tensor theory formulated by Sen and Dunn when the scalar field introduced in the theory is independent of time. This may, possibly, be due to the fact that the interaction of the time-dependent scalar field with the electromagnetic field stimulates electromagnetic monopole radiation.

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